INEQUALITIES RELATED TO THE S-DIVERGENCE

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Abstract. The S-Divergence is a distance like function on the convex cone of positive definite matrices, which is motivated from convex optimization. In this paper, we will prove some inequalities for Kubo-Ando means with respect to the square root of the S-Divergence.

1. Introduction

Let $\mathbb{H}_n$ denote the set of all $n \times n$ Hermitian matrices. The set of all positive definite (henceforth *positive*) matrices in $\mathbb{H}_n$ is denoted by $\mathbb{P}_n$. The *Frobenius norm* of a matrix A is $\|A\|_F = \sqrt{\operatorname{tr}(A^*A)}$, while $\|A\|$ denoted the operator norm.

The set $\mathbb{P}_n$ is a well-studied differentiable Riemannian manifold, with the Riemannian metric given by the differential form $ds = \|A^{-1/2}dAA^{-1/2}\|_F$. The metric induces the *Riemannian distance* (for more information, one can see, e.g., [2, Chapter 6]):

$$\delta_R(A,B) := \|\log(B^{-1/2}AB^{-1/2})\|_F, \quad \forall\, A,B>0. \tag{1.1}$$

Motivated from convex optimization, one can define the *S-Divergence*:

$$\delta_S^2(A,B) = \log\det\left(\frac{A+B}{2}\right) - \frac{1}{2}\log\det(AB),\ \forall\, A,B>0. \tag{1.2}$$

Sra exhibited several properties akin to the Riemannian distance δ_R (see [15]). Note that the S-divergence δ_S^2 is non-negative definite and symmetric, but not a *metric*. Indeed, Sra prove that δ_S is a metric on $\mathbb{P}_n$ (see [15, Theorem 3.1]).

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Note that the equality $\log\det A = \operatorname{Tr}\log A$ holds for all $A \in \mathbb{P}_n$, by the argument of [11, p.28], we have that

$$\begin{aligned} \delta_S^2(A,B) &= \log\det\Big(\frac{A^{-1/2}BA^{-1/2}+I}{2}\Big) - \frac{1}{2}\log\det(A^{-1/2}BA^{-1/2}) \\ &= \operatorname{Tr}\Big[\log\Big(\frac{A^{-1/2}BA^{-1/2}+I}{2}\Big) - \log(A^{-1/2}BA^{-1/2})^{1/2}\Big]. \end{aligned} \tag{1.3}$$

It follows that for any $\lambda > 0$, we have that $\delta_S(\lambda A, \lambda B) = \delta_S(A,B)$.

In this paper, we study the isometries with respect to δ_S in Section 2, which improves Molnár's result [11, Theorem 4]. In section 3, we prove some inequalities related to the geometric mean, spectral geometric mean and Wasserstein mean under the S-divergence.

2. Isometries with respect to δ_S

Molnár gave the structures of isometries on the metric space $(\mathbb{P}_n, \delta_S)$ as follows:

Theorem 2.1. *(see [11, Theorem 4]) Assume $n \geq 2$. Suppose that $\phi : \mathbb{P}_n \to \mathbb{P}_n$ is a bijective map such that*

$$\delta_S(\phi(A), \phi(B)) = \delta_S(A,B), \quad \forall\, A, B \in \mathbb{P}_n. \tag{2.1}$$

Then there is an invertible matrix $T \in M_{n\times n}$ such that ϕ is of one of the following forms:

(s1) $\phi(A) = TAT^*$,
(s2) $\phi(A) = TA^{-1}T^*$,
(s3) $\phi(A) = TA^{tr}T^*$,
(s4) $\phi(A) = T(A^{tr})^{-1}T^*$

for all $A \in \mathbb{P}_n$.

Actually, we can improve the above result. Let $\mathbb{S}_n$ be the set of all $n \times n$ positive matrices with unit trace.

Theorem 2.2. *Suppose that $\phi : \mathbb{S}_n \to \mathbb{S}_n$ be a bijective isometry with respect to δ_S, then there is an invertible matrix $T \in M_{n\times n}$ such that ϕ is of one of the following forms:*

(s1) $\phi(A) = TAT^*$,
(s2) $\phi(A) = TA^{-1}T^*$,
(s3) $\phi(A) = TA^{tr}T^*$,
(s4) $\phi(A) = T(A^{tr})^{-1}T^*$

for all $A \in \mathbb{S}_n$.

Proof. By the assumption of ϕ, one can define $\psi : \mathbb{P}_n \to \mathbb{P}_n$ by

$$\psi(A) = \mathrm{tr}(A)\phi(\frac{A}{\mathrm{tr}(A)}), \quad \forall\, A \in \mathbb{P}_n.$$

Then it is easy to see that ψ is a bijective and

$$\delta_S(\psi(A), \psi(B)) = \delta_S(A, B)$$

for any $A, B \in \mathbb{P}_n$ with $\mathrm{tr}(A) = \mathrm{tr}(B)$. Moreover, conditions in [11, Proposition 8] are filfilled and then, by the proof of [11, Lemma 9], we have that

$$\psi(AB^{-1}A) = \psi(A)\psi(B)^{-1}\psi(A), \quad \forall\, A, B \in \mathbb{P}_n.$$

By the similar argument in [11, Theorem 4], one can find an invertible matrix $T \in M_{n\times n}$ such that ψ is of one of the following forms:

(s1) $\psi(A) = TAT^*$,
(s2) $\psi(A) = TA^{-1}T^*$,
(s3) $\psi(A) = TA^{tr}T^*$,
(s4) $\psi(A) = T(A^{tr})^{-1}T^*$

for all $A \in \mathbb{P}_n$. In particular, ϕ must be of the form required. □

3. Inequalities related to various means

In this section, we will prove some inequalities related to some Kubo-Ando means. For positive matrices A and B, recall that the *geometric mean* $A\sharp B$ is defined by

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

The geometric mean has a lot of attractive properties (see, e.g., [1, 9]). Surprisingly, Sra proved the following result

Theorem 3.1. [15, Theorem 4.1] *$A\sharp B$ is the equidistant from A and B, that is,*

$$\delta_S(A, A\sharp B) = \delta_S(B, A\sharp B).$$

Suppose that $t \in [0, 1]$, then one can define the *Wasserstein mean* of $A, B \in \mathbb{P}_n$ by

$$\begin{aligned} A \diamond_t B &= (1-t)^2A + t^2B + t(1-t)[A^{1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2} \\ &\quad + A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{1/2}] \\ &= (1-t)^2A + t^2B + t(1-t)[(AB)^{1/2} + (BA)^{1/2}] \\ &= A^{-1/2}[(1-t)A + t(A^{1/2}BA^{1/2})^{1/2}]^2A^{-1/2}. \end{aligned}$$

Bhatia, Jain and Lim [3, p.180] proved that $A \diamond_t B$ is the natural parametrisation of the geodesic joining A and B.

Theorem 3.2. *For any $A, B \in \mathbb{P}_n$ and any $t \in (0,1)$, we have that*

$$\delta_S^2(A, A \diamond_t B) \geq 2\delta_S^2(I, (1-t)I + tA^{-1}\sharp B).$$

Proof. Let $C = A^{1/2}BA^{1/2}$. By [15, Theorem 4.5 and Corollary 4.10], we can derive that

$$\begin{aligned} & \delta_S^2(A, A \diamond_t B) \\ = \ & \delta_S^2(A^2, [(1-t)A + t(A^{1/2}BA^{1/2})^{1/2}]^2) \\ \geq \ & 2\delta_S^2(A, (1-t)A + t(A^{1/2}BA^{1/2})^{1/2}) \\ = \ & 2\delta_S^2(I, (1-t)I + tA^{-1}\sharp B). \end{aligned}$$

□

Remark 3.3. For A and B, when put $C = A^{1/2}BA^{1/2}$, we just can prove that

$$\begin{aligned} & \delta_S^2(B, A \diamond_t B) \\ = \ & \delta_S^2(C, ((1-t)A + tC^{1/2})^2) \\ = \ & 2\delta_S^2(C^{1/2}, (1-t)A + tC^{1/2}). \end{aligned}$$

Moreover, one can define the *spectral geometric mean* between positive matrices A and B:

$$A\natural B = (A^{-1}\sharp B)^{1/2}A(A^{-1}\sharp B)^{1/2}$$

(we refer [9] for more details). It is easy to see that $\delta_S^2(A^{-1}\sharp B, A\natural B) = \delta_S^2(I, A)$.

Proposition 3.4. *For any positive matrices A and B, we have that*

$$\delta_S^2(I, A\natural B) \leq \frac{1}{2}\delta_S^2(B, A^{-1}).$$

Proof. By the definition, one can derive that

$$\begin{aligned} \delta_S^2(I, A\natural B) &= \delta_S^2((A^{-1}\sharp B)^{-1}, A) = \delta_S^2(A^{-1}\sharp B, A^{-1}) \\ &= \delta_S^2((A^{1/2}BA^{1/2})^{1/2}, I) \\ &\leq \frac{1}{2}\delta_S^2(A^{1/2}BA^{1/2}, I) \\ &= \frac{1}{2}\delta_S^2(B, A^{-1}). \end{aligned}$$

□

More generally, one can define weighted spectral geometric mean (see, e.g., [10]). For $0 \leq t \leq 1$. Let A, B be positive matrices, the *weighted spectral geometric mean* is defined by

$$A\natural_t B = (A^{-1}\sharp B)^t A(A^{-1}\sharp B)^t.$$

By the definition, it is easy to prove the following properties:

Lemma 3.5. *For any $s, t \in [0,1]$ and any positive matrices A, B, we have that*

(i) *if $t > s$, then $\delta_S^2(A\natural_s B, A\natural_t B) = \delta_S^2(A, A\natural_{t-s} B)$;*
(ii) *if $t < s$, then $\delta_S^2(A\natural_s B, A\natural_t B) = \delta_S^2(A\natural_{s-t} B, A)$;*

When $1/2 < t < 1$, we have

$$\begin{aligned} & \delta_S^2(A^{-1}\sharp B, A\natural_t B) \\ = \;& \delta_S^2(I, (A^{-1}\sharp B)^{t-1/2} A (A^{-1}\sharp B)^{t-1/2}) \\ = \;& \delta_S^2(I, A\natural_{t-1/2} B). \end{aligned}$$

On the other hand, to give a universal esitimate, we can prove the following inequality.

Theorem 3.6. *If $t \neq 1/2$, for any positive matrices A, B, we have*

$$\delta_S^2(A^{-1}\sharp B, A\natural_t B) \leq \frac{|1-2t|}{2} \delta_S^2(B, A^{(3-2t)/(1-2t)}).$$

Proof. When $0 < t < 1/2$, it follows from the properties of S-divergence δ_S that

$$\begin{aligned} & \delta_S^2(A^{-1}\sharp B, A\natural_t B) \\ = \;& \delta_S^2((A^{-1}\sharp B)^{1-2t}, A) \\ \leq \;& (1-2t)\delta_S^2(A^{-1}\sharp B, A^{1/(1-2t)}) \\ = \;& (1-2t)\delta_S^2((A^{1/2}BA^{1/2})^{1/2}, A^{1+1/(1-2t)}) \\ \leq \;& \frac{1-2t}{2}\delta_S^2(A^{1/2}BA^{1/2}, A^{(4-4t)/(1-2t)}) \\ = \;& \frac{1-2t}{2}\delta_S^2(B, A^{(4-4t)/(1-2t)-1}) \\ = \;& \frac{1-2t}{2}\delta_S^2(B, A^{(3-2t)/(1-2t)}). \end{aligned}$$

When $1/2 < t < 1$, by a similar argument,

$$\begin{aligned}
& \delta_S^2(A^{-1}\sharp B, A\natural_t B) \\
= \;& \delta_S^2((A^{-1}\sharp B)^{1-2t}, A) \\
= \;& \delta_S^2((A^{-1}\sharp B)^{2t-1}, A^{-1}) \\
\leq \;& (2t-1)\delta_S^2(A^{-1}\sharp B, A^{1/(1-2t)}) \\
= \;& (2t-1)\delta_S^2((A^{1/2}BA^{1/2})^{1/2}, A^{1+1/(1-2t)}) \\
\leq \;& \frac{2t-1}{2}\delta_S^2(A^{1/2}BA^{1/2}, A^{(4-4t)/(1-2t)}) \\
= \;& \frac{2t-1}{2}\delta_S^2(B, A^{(4-4t)/(1-2t)-1}) \\
= \;& \frac{2t-1}{2}\delta_S^2(B, A^{(3-2t)/(1-2t)}).
\end{aligned}$$

□

Remark 3.7. We also can derive that

$$\delta_S^2(A^{-1}\sharp B, A\natural_t B) = \delta_S^2((A^{-1}\sharp B)^{1-2t}, A).$$

Remark 3.8. Note that $A\natural_t B$ is the solution of the equation $(A^{-1}\sharp B)^t = A^{-1}\sharp X$, then we have that

$$\begin{aligned}
& \delta_S^2(A, A\natural_t B) \\
= \;& \delta_S^2(A^{1/2}AA^{1/2}, A^{1/2}(A\natural_t B)A^{1/2}) \\
\geq \;& 2\delta_S^2(A, (A^{1/2}(A\natural_t B)A^{1/2})^{1/2}) \\
= \;& 2\delta_S^2(I, A^{-1}\sharp(A\natural_t B)) \\
= \;& 2\delta_S^2(I, (A^{-1}\sharp B)^t).
\end{aligned}$$

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